Value-distribution Theory for Holomorphic Almost Periodic Functions

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1. *Introduction*

I am deeply indebted to the organizers for inviting me to participate in this conference on the occasion of the 100$^{\text{th}}$ anniversary of Harald Bohr.

My student years here in Copenhagen happened to coincide with the last 5 years of Bohr's life. Although many years have passed I still have a vivid memory of his inspiring lectures and of his personal kindness.

Today I am going to talk about some work of my own from this time; this was a response to a prize question posed by the University for 1950 concerning holomorphic almost periodic functions. This work was inspired by papers by Bohr and by Jessen, together with the works of the Finnish mathematical school (primarily Rolf Nevanlinna and Lars Ahlfors) on value-distribution theory of meromorphic functions. The new results are described in §§ 4-5.

First I will describe some background material. Here I am indebted to Prof. B. Fuglede and Prof. H. Tornehave for some informative references. Bohr's early work on Dirichlet series and the Riemann zeta-function led him to the theory of almost periodic functions. While the principal results of his theory of almost periodic functions on **R** have to some extent been absorbed in the theory of continuous functions on compact abelian groups, his theory of holomorphic almost periodic functions [2] has retained its independence and its charm.

A holomorphic function $f(s)$ in a vertical strip (α, β): $\alpha < \operatorname{Re} s < \beta$ is said to be *almost periodic* if to each $\epsilon > 0$ there exists a number $l = l(\epsilon)$ such that each interval $t_0 < t < t_0 + l$ of length l contains a number τ such that

$$|f(s+i\tau) - f(s)| \leq \epsilon$$

for all s in the strip. (Here α and β are allowed to be infinite.) In other words, if $s = \sigma + it$, $\alpha < \sigma < \beta$, the function $t \to f(\sigma + it)$ is almost periodic on **R** and uniformly so for $\alpha < \sigma < \beta$.

To each such function f one can associate its Dirichlet series

$$f(s) \sim \sum_n A_n e^{\Lambda_n s} \qquad \Lambda_n \in \mathbf{R}, \tag{1}$$

which determines it uniquely. Here

$$A_n = \mathcal{M}_t\left(f(\sigma + it)e^{-\Lambda_n(\sigma + it)}\right) \tag{2}$$

where $\mathcal{M}$ is the mean value

$$\mathcal{M}(\varphi) = \lim_{S-R\to\infty} \frac{1}{S-R} \int_R^S \varphi(t)dt.$$

For φ almost periodic this limit does indeed exist and then the holomorphy of f implies that A_n is indeed independent of σ. A uniformly convergent Dirichlet series (say $\varsigma(s) = \Sigma_1^\infty e^{-(\log n)s}$ for Re $s > 1 + \epsilon$) is almost periodic; on the other hand, to an almost periodic function $f(s)$ in a strip can be associated a sequence $f_p(s)$ of exponential polynomials $\Sigma_n A_n^{(p)} \exp(\Lambda_n^{(p)} s)$ which converge to $f(s)$ uniformly in any closed substrip ($\alpha_1 \leq$ Re $s \leq \beta_1$, where $\alpha < \alpha_1 < \beta_1 < \beta$).

The original Dirichlet series

$$\sum_1^\infty \frac{a_n}{n^s} = \sum_1^\infty a_n e^{-(\log n)s}$$

were generalized to series of the form

$$\sum_n a_n e^{\Lambda_n s} \qquad \Lambda_1 > \Lambda_2 > \ldots \tag{3}$$

and both at the beginning and the end of his career Bohr investigated problems of convergence, summability etc. for such series (3). It is therefore worth stressing that in (1) the order of the exponents is unrestricted.

2. *Result of Jessen. The Jensen Function*

With the Riemann zeta-function as motivation it becomes a problem of interest to study the distribution of zeros of a function $f(s)$ almost periodic in a strip (α, β). For such functions f the basic general results were obtained by Jessen [6]. He showed the existence of the limit

$$\varphi_f(\sigma) = \mathcal{M}_t(\log|f(\sigma + it)|) = \lim_{S-R\to\infty} \frac{1}{S-R} \int_R^S \log|f(\sigma + it)|\,dt \tag{1}$$

(in spite of the fact that f may have zeros) and proved that it is a *convex* function of σ. Jessen's principal result is the following theorem. If $\alpha < \alpha' < \beta' < \beta$ let $n(\alpha', \beta'; R, S)$ denote the number of zeros of f in the rectangle $\alpha' < \sigma < \beta'$, $R < t < S$ counted with multiplicity.

THEOREM 2.1. *If φ_f is differentiable at α' and β' then the density of zeros*

$$H(\alpha', \beta') = \lim_{S-R\to\infty} \frac{n(\alpha', \beta'; R, S)}{S-R}$$

exists and

$$H(\alpha', \beta') = \frac{1}{2\pi}\{\varphi_f'(\beta') - \varphi_f'(\alpha')\}.$$

If the function $t \to f(\sigma + it)$ has a fixed period p it turns out that this is equivalent to the classical Jensen formula in complex function theory; for this case the function φ_f is a piecewise linear function. Jessen called φ_f *the Jensen function* for f.

Indication of proof. First we assume that the boundary of the rectangle $\alpha' < \sigma < \beta'$, $R < t < S$ contains no zero of $f(s)$. Then by standard complex variable theory.

$$2\pi\, n(\alpha', \beta'; R, S) =$$
$$\int_R^S \frac{f'(\beta' + it)}{f(\beta' + it)}dt - \int_R^S \frac{f'(\alpha' + it)}{f(\alpha' + it)}dt - i\int_{\alpha'}^{\beta'} \frac{f'(\sigma + iR)}{f(\sigma + iR)}d\sigma + i\int_{\alpha'}^{\beta'} \frac{f'(\sigma + iS)}{f(\sigma + iS)}d\sigma.$$

Consider the vertical segments $\alpha' + it$ $(R \le t \le S)$ and $\beta' + it$ $(R \le t \le S)$. We can find a simply connected region Ω containing both of these segments and no zeros for f. We can then define the logarithm $\log f(s)$ in Ω, divide the relation above by $S-R$ and let it tend to ∞. We can restrict the R and S in such a way that the two last terms above give no contribution in the limit. The identity in Theorem 2.1 follows by taking real parts. The restriction on α' and β' is then removed by a continuity argument.

It is now an interesting problem to characterize the convex functions $\varphi(\sigma)$ which arise as Jensen functions φ_f for suitable almost periodic $f(s)$. This question was investigated by Buch [4] whose results imply for example that any convex function which is not linear on any interval arises in this fashion. A complete characterization of the φ_f was given by Jessen and Tornehave [7], § 112. It implies for example that a convex function $\varphi(\sigma)$, $\alpha < \sigma < \beta$, having infinitely many intervals of linearity in a compact subinterval of (α, β) cannot be a Jensen function φ_f if the slopes $\varphi'(\sigma)$ in these intervals are linearly independent over the rational numbers.

3. *Normal Almost Periodic Functions*

Already in his original paper [2] Bohr made a special investigation of almost periodic functions $f(s)$ in a half-plane $(-\infty, \beta)$ and expressed their behaviour near $\sigma = -\infty$ in terms of their Dirichlet expansion

$$f(s) \sim \sum_n A_n e^{\Lambda_n s} \qquad (A_n \neq 0). \tag{1}$$

In [3] he singled out the so-called *normal almost periodic functions* $f(s)$ for which among the nonzero exponents Λ_n the smallest one exists. These have the following property:

Given any $a \in \mathbf{C}$, *there exists a half-plane* $(-\infty, \sigma_a)$ *which contains no a-point for* $f(s)$ (*i.e. a zero of* $f(s) - a$).

Let us for a moment view such a function $f(s)$ via the substitution $s = \log z$ as a function $\varphi(z)$ on a piece $0 < \rho < \rho_0$ of the Riemann surface of $\log z$. The series (1) then becomes a generalized Laurent series

$$\varphi(z) \sim \Sigma A_n z^{\Lambda_n} \qquad A_n \neq 0. \tag{2}$$

Let us for simplicity assume the normalizing property that the lowest nonzero exponent, say Λ_0, is > 0. Bohr showed in [3] that the inverse function is also normal almost periodic. I have proved in [5] that a similar statement can be made about the composition of two normal almost periodic functions (having the above normalizing property).

4. *Value-distribution Theory. The first Fundamental Theorem*

Consider a fixed $\beta \leq \infty$ and let $z = f(s)$ be normal almost periodic in $\{-\infty, \beta\}$ (that is normal almost periodic in any substrip $(-K, \beta_1)$ where $\beta_1 < \beta$). We apply stereographic projection of the z-plane $\mathbf{C} \cup \{\infty\}$ onto the Riemann sphere S with diameter 1, tangential to the z-plane at $z = 0$. Given $z_1, z_2 \in \mathbf{C} \cup \{\infty\}$ the (chordal) distance of the corresponding points on S is given by

$$k(z_1,z_2) = \frac{|z_1 - z_2|}{\sqrt{(1+|z_1|^2)\,(1+|z_2|^2)}}$$

so the arc-length element $d\sigma$ on S is

$$d\sigma = \frac{|dz|}{1+|z|^2}.$$

For any $a \in \mathbf{C} \cup \{\infty\}$ let Λ_a denote the smallest exponent in the Dirichlet expansion of $f(s) - a$. (Here $f(s) - \infty$ is to be understood as $1/f(s)$ which is also normal almost periodic). Let A_a denote the corresponding coefficient. We now introduce a quantity which measures how well the function $f(s)$ approximates the value a on the line $\mathrm{Re}\, s = \sigma$. Put

$$M(\sigma,a) = \mathscr{M}_t\{-\log(k(f(\sigma+it), a))\} + [\log k(f(-\infty),a)], \tag{1}$$

where the remainder term is

$$[\log k(f(-\infty),a)] = \begin{cases} \log k(f(-\infty),a), & f(-\infty) \neq a \\ \log \dfrac{|A_a|}{1+|a|^2} & f(-\infty) = a \neq \infty \\ \log\ |A_\infty| & f(-\infty) = a = \infty. \end{cases}$$

The existence of the integral in (1) is clear from the existence of (1) § 2. Next we put $n(\sigma,a;R,S)$ = the number of a-points (with multiplicity) of $f(s)$ in the rectangle $-\infty < \tau < \sigma$, $R < t < S$ (with $s = \tau + it$).

$$N(\sigma,a) = \lim_{S-R\to\infty} \frac{2\pi}{S-R}\int_{-\infty}^{\sigma} n(\tau,a;R,S)\, d\tau + n(-\infty,a;-\infty,\infty)\sigma, \tag{2}$$

where

$$n(-\infty,a;-\infty,\infty) = \mathrm{Max}(\Lambda_a,0).$$

The existence of the last limit is easily established by means of tools used in the proof of Theorem 2.1. The function $N(\sigma,a)$ is taken as a measure for the number of a-points of f in the half-plane $\mathrm{Re}\, s < \sigma$. Note that the remainder term in (2) appears only if $a = \lim_{\sigma\to-\infty} f(s)$.

Theorem 4.1. *If $f(s)$ is normal almost periodic in $\{-\infty, \beta\}$ then the sum*

$$M(\sigma,a) + N(\sigma,a) = T(\sigma) \tag{3}$$

is independent of a. Also

$$T(\sigma) = \lim_{S-R\to\infty} \frac{2\pi}{S-R}\int_{-\infty}^{\sigma} A_{RS}(\tau)d\tau, \tag{4}$$

where $A_{RS}(\tau)$ *is the area of the Riemann surface* $V_{R,S}(\tau)$ *over the Riemann sphere onto which the function* f *maps the rectangle* $-\infty < \rho < \tau$, $R < t < S$.

The fact that the sum $M(\sigma, a) + N(\sigma, a)$ is independent of a is an analog of Nevanlinna's first fundamental theorem for meromorphic functions. It implies that if $N(\sigma,a)$ is small that is, if $f(s)$ has few a-points, then the approximation term $M(\sigma,a)$ is large and vice-versa. The function $T(\sigma)$ is called *the characteristic function.* The geometric interpretation (4) of $T(\sigma)$ is an analog of a similar interpretation for the classical (periodic) case given by Ahlfors [1] and Shimizu [9].

The proof of Theorem 4.1 proceeds along lines similar to the classical theory (Nevanlinna [8], VI, § 3) but requires in addition some tools utilized in the proof of Theorem 2.1. A brief indication follows. Let $A \in \mathbf{C}$ and put

$$w(s) = A + \frac{1}{f(s)}, \qquad v(s) = \log(1 + |w(s)|^2).$$

We use Gauss' formula

$$\int_\Gamma \frac{\partial v}{\partial n}\, dl = \iint_\Omega \Delta v \; d\tau dt \tag{5}$$

on a region Ω which is the rectangle $\sigma_0 < \tau < \sigma$, $R < t < S$ with small disks removed having the zeros of f as the centers. Here Γ is the boundary of Ω (with the appropriate orientation), dl the arc element on Γ, Δ the Laplacian in the (τ,t) variables and $\partial/\partial n$ the outgoing normal derivative.

The proof now proceeds along the following steps.

(i) We use the Laurent series of $1/f(s)$ around each zero of $f(s)$ to estimate the contributions to the left hand side of (5) of the circular parts of Γ. Then we let the radii of the disks considered tend to 0.

(ii) By direct computation

$$(\Delta v)(s) = 4\frac{|w'(s)|^2}{(1+|w(s)|^2)^2}.$$

Viewing $w(s)$ as a map from the s-plane to the Riemann sphere lying on the w-plane we have

$$\frac{d\sigma}{|ds|} = \frac{1}{1+|w(s)|^2}|w'(s)|.$$

Thus if $A_{RS}(\tau)$ is the area function (for the function $w(s)$) we have (with $s = \tau + it$)

$$A_{RS}(\sigma) = \int_R^S dt \int_{-\infty}^{\sigma} \frac{|w'(s)|^2}{(1+|w(s)|^2)^2} d\tau.$$

(iii) In (5) we divide by $S-R$ and let $S-R \to \infty$ through special values of S and R, such that on the corresponding horizontal lines $\tau + iR$, $\tau + iS$ ($\sigma_0 < \tau < \sigma$) $f(s)$ is bounded away from 0. Then the horizontal pieces of the boundary Γ in (5) give no contribution to the limit. The normal derivatives in (5) can be pulled outside the integral as $\partial/\partial\sigma$.

(iv) Now let $\sigma_0 \to -\infty$ in (5) and then integrate with respect to σ from $-\infty$ to σ. Considering the behaviour of $w(s)$ as $\sigma \to -\infty$ we obtain after some manipulation the formula

$$\begin{aligned} &\mathcal{M}_t \left(\log \sqrt{1+|w(\sigma+it)|^2}\right) + N(\sigma,0) = \\ &= \lim_{S-R\to\infty} \frac{2}{S-R} \int_{-\infty}^{\sigma} A_{RS}(\tau)d\tau + \log \sqrt{1+|w(-\infty)|^2}. \end{aligned} \tag{6}$$

The last term should be replaced by $\log(1/|A_0|)$ in case $w(-\infty) = \infty$.

(v) Consider a fixed $a \in \mathbf{C}$ and the function

$$w_1(s) = \frac{1+\bar{a}f(s)}{f(s)-a} = \bar{a} + \frac{1}{(f(s)-a)\,(1+|a|^2)^{-1}}.$$

The values of $w_1(s)$ are obtained from the values of $f(s)$ by rotation of the sphere so A_{RS} is the same for w_1 and for f. Also

$$(1 + |w_1(s)|^2)^{-1} = k(w_1(s), \infty) = k(f(s),a)$$

so when (6) is used on w_1 we do obtain Theorem 4.1.

5. *The Second Fundamental Theorem. Applications*

While the first fundamental theorem expresses the constancy of the *total affinity* $M(\sigma,a) + N(\sigma,a)$ of $f(s)$ to the value a the second fundamental theorem will show that for most a $N(\sigma,a)$ is the principal component. This is based on an estimate of the sum $\sum_{v=1}^{p} M(\sigma,a_v)$ for arbitrary distinct $a_1, \dots a_p$.

Motivated by the classical (periodic) theory we consider the number $n_1(\sigma, a; R, S)$ of multiple roots in the equation $f(s) - a = 0$ in $-\infty < \tau < \sigma$, $R < t < S$, such that a k-fold root is only counted $(k-1)$ times. We also put $n_1(-\infty, a; -\infty, \infty) = \mathrm{Max}(\Lambda'_a, 0)$, where Λ'_a is the smallest exponent in the expansion of $f'(s) - a$. Clearly $\sum_a n_1(\sigma, a;\ R, S)$ is the number of zeros for $f'(s)$ in the rectangle indicated. Then we put

$$N_1(\sigma,a) = \lim_{S-R\to\infty} \frac{2\pi}{S-R}\int_{-\infty}^{\sigma} n_1(\tau,a;\ R,S)d\tau + n_1(-\infty,\ a;-\infty,\infty)\sigma\ , \tag{7}$$

and by the remark above, $\sum_a N_1(\sigma,a)$ is bounded by the function $N(\sigma, 0)$ taken for the derivative f'. In the next theorem (the analog of Nevanlinna's second fundamental theorem) we distinguish between the two cases: β finite and $\beta = \infty$.

Theorem 5.1. I. *Let $f(s)$ be normal almost periodic in $\{-\infty,\infty\}$ and $a_1, \dots, a_p$ arbitrary distinct complex numbers. The inequality*

$$\sum_{\nu=1}^{p} M(\sigma,\ a_\nu) + \sum_a N_1(\sigma,a) \leq T(\sigma) + O(\log\ T(\sigma)) + O(\log|\sigma|) \tag{8}$$

holds for all σ except on a set of σ of finite measure.
II. *Let $f(s)$ be normal almost periodic in $\{-\infty, 0\}$ and $a_1, \dots, a_p$ any distinct complex numbers. Then inequality* (8) *holds with* $\log|\sigma|$ *replaced by* $\log(1/(1-e^{\sigma}))$ *and the inequality holds for all $\sigma < 0$ except for at set of σ over which the integral of $e^{\sigma}(1-e^{\sigma})^{-1}$ is finite.*

In the proof of this theorem the passage from periodic functions to almost periodic functions gives rise to certain technical difficulties. The proof is therefore too complicated to describe here in detail. Instead, I will show how the theorem implies the analog of Nevanlinna's defect relation.

Application. Let f be normal almost periodic in $\{-\infty, \beta\}$. For each $a \in \mathbf{C}$ we define the *defect* by

$$\delta(a) = 1 - \overline{\lim_{\sigma\to\beta}}\ \frac{N(\sigma,a)}{T(\sigma)}$$

and the *ramification index*

$$\nu(a) = \lim_{\sigma\to\beta} \frac{N_1(\sigma,a)}{T(\sigma)}\ .$$

In the case when f is a nonconstant normal almost periodic function in $\{-\infty, \infty\}$ it is easily deduced from Theorem 4.1 that $\underline{\lim}_{\sigma\to\infty}\ T(\sigma)/\sigma > 0$. From Theorem 5.1 we can therefore deduce the following result.

THEOREM 5.2. Let *$f(s)$ be nonconstant and normal almost periodic in $\{-\infty, \infty\}$. Then the defect $\delta(a)$ and the ramification index $\nu(a)$ are strictly positive for at most countably many a and*

$$\sum_a \delta(a) + \sum_a \nu(a) \leq 1.$$

The defect $\delta(a)$ is a measure for how rarely f takes the value a. If a is omitted by $f(s)$ altogether in $-\infty \leq \operatorname{Re} s < \infty$ then $\delta(a) = 1$ so we deduce from $\sum_a \delta(a) \leq 1$ that there can be at most one such value a.

For the case $\beta = 0$ we obtain similarly the following result.

THEOREM 5.3. *Let $f(s)$ be normal almost periodic in $\{-\infty, 0\}$ and assume*

$$\varlimsup_{\sigma \to 0} \frac{\log\left(\frac{1}{1-e^{\sigma}}\right)}{T(\sigma)} = 0.$$

Then

$$\sum_a \delta(a) \leq 1.$$

Again this implies that f omits at most one value in the strip $-\infty \leq \operatorname{Re} s < 0$.

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